Stability = Collapse!
Revisit of Hua's Theorem On Input-Output Models

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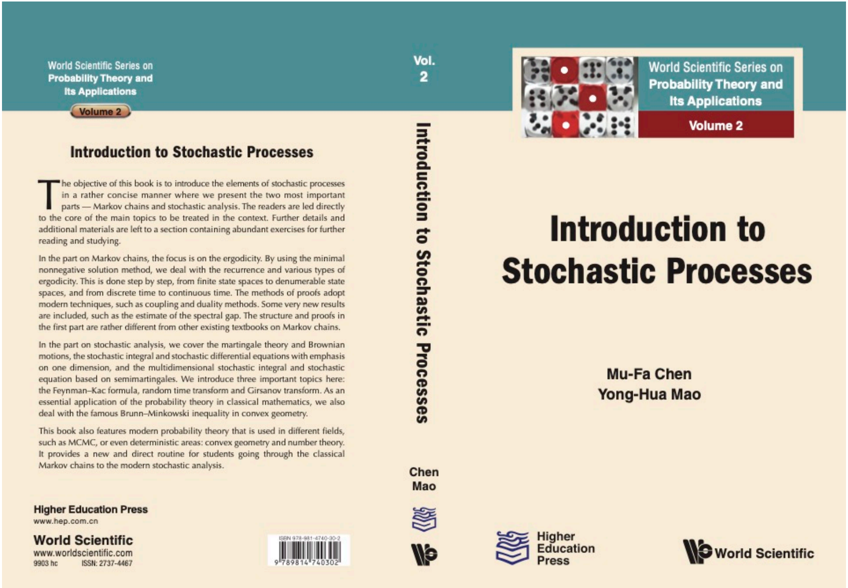
Book


World Scientific Series on Probability Theory and Its Applications
Volume 2
Introduction to Stochastic Processes
The objective of this book is to introduce the elements of stochastic processes in a rather concise manner where we present the two most important parts — Markov chains and stochastic analysis. The readers are led directly to the core of the main topics to be treated in the context. Further details and additional materials are left to a section containing abundant exercises for further reading and studying.
In the part on Markov chains, the focus is on the ergodicity. By using the minimal nonnegative solution method, we deal with the recurrence and various types of ergodicity. This is done step by step, from finite state spaces to denumerable state spaces, and from discrete time to continuous time. The methods of proofs adopt modern techniques, such as coupling and duality methods. Some very new results are included, such as the estimate of the spectral gap. The structure and proofs in the first part are rather different from other existing textbooks on Markov chains.
In the part on stochastic analysis, we cover the martingale theory and Brownian motions, the stochastic integral and stochastic differential equations with emphasis on one dimension, and the multidimensional stochastic integral and stochastic equation based on semimartingales. We introduce three important topics here: the Feynman–Kac formula, random time transform and Girsanov transform. As an essential application of the probability theory in classical mathematics, we also deal with the famous Brunn–Minkowski inequality in convex geometry.
This book also features modern probability theory that is used in different fields, such as MCMC, or even deterministic areas: convex geometry and number theory. It provides a new and direct routine for students going through the classical Markov chains to the modern stochastic analysis.
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Input-output model

- **Vector of products** $(x^{(1)}, x^{(2)}, \cdots, x^{(d)})$: the quantity of the main products we are interested in.
- The initial vector of products (the input last year), denote by

$$x_0 = (x_0^{(1)}, x_0^{(2)}, \cdots, x_0^{(d)}).$$

- The output one year later:

$$x_1 = (x_1^{(1)}, x_1^{(2)}, \cdots, x_1^{(d)}).$$

- The structure matrix $A = (a_{ij})$: to produce one unit of ith product, one needs a_{ij} units of the jth product. Thus the output x_1 expends $\sum_{i=1}^{d} x_1^{(i)} a_{ij}^{(0)}$ units of the jth product. This is the input of the jth product last year.

Equation

- Therefore $x_0^{(j)} = \sum_{i=1}^{d} x_1^{(i)} a_{ij}$ or in the vector form:

$$x_0 = x_1 A.$$

- Year after year: $x_{n-1} = x_n A$. Thus

$$x_0 = x_1 A = x_2 A^2 = \cdots = x_n A^n.$$

- This is called the Input-output (IPOP) equation.

Question

- The question is how to develop economy well, or how fast the economy can be?
- From the IPOP equation, the question becomes: how to choose x_0 to make x_n grow as fast as possible? That is, we expect the ratios $x_1^{(j)}/x_0^{(j)}$ are as great as possible.
- The **minimax principle**: finding out the best solution among the worst cases. This is the safest strategy, and used widely in the optimization theory and the game theory.
- Namely, for a fixed structure matrix A, find out x_0 such that

$$\max_{\substack{x_1>0 \\ x_0=x_1A}} \min_{1\leq j\leq d} \frac{x_1^{(j)}}{x_0^{(j)}}$$

attains the maximum.

L. K. Hua's fundamental theorem

Theorem

Assume A is invertible, nonnegative and irreducible. Let u be the left eigenvector corresponding to its largest eigenvalue $\rho = \rho(A)$

(1) *If $x_0 = u$ (up to a positive factor), then $x_n = x_0\rho^{-n}$, $n \geq 1$. In this case, it has the greatest increasing rate ρ^{-1}.*

(2) *If $0 < x_0 \neq u$ (up to a positive factor) and assume additionally A^{-1} is not nonnegative, then there exist n_0 and j_0 such that $x_{n_0}^{(j_0)} \leq 0$. In this case, the economy tends to collapse.*

"Collapse" means that in some year, x_n contains a negative component, that is the **collapse time**

$$T := \inf\{n \geq 1 : \text{ there exists some } j \text{ such that } x_n^{(j)} < 0\}$$

is finite.

A basic example

- Consider two products only: industry and agriculture.
- Let

$$A = \frac{1}{100}\begin{pmatrix} 25 & 14 \\ 40 & 12 \end{pmatrix}.$$

Then

$$u = \left(\frac{5}{7}(\sqrt{2409}+13),\ 20\right) \approx (44.34397483, 20).$$

For different inputs, the collapse times are listed in the following table.

x_0	T
$(44, 20)$	3
$(44.344, 20)$	8
$(44.34397483, 20)$	13

- This shows that the economy is rather sensitive!

Extreme case: Stability = Collapse!

- Let A be a positive rank one matrix:

$$a_{ij} = v_i u_j > 0, \quad \text{or} \quad A = vu$$

with v a column vector and u a row vector satisfying $uv = 1$

- Now, by assuming $x_0 v = 1$, we have

$$1 = x_0 v = x_1 A v = (x_1 v)(uv) = x_1 v$$

and

$$x_0 = x_1 A = (x_1 v) u = u.$$

- If $x_0 \neq u$, then $T = 1$!
- Stability = Collapse!

Perron(1907)-Frobenius(1912) theorem

Theorem

The greatest eigenvalue $\rho = \rho(A)$ *for the irreducible nonnegative matrix* A *is positive, and its corresponding left* $($*row*$)$ *vector* u *and right* $($*column*$)$ *eigenvector* v *are positive:*

$$uA = \rho u, \quad Av = \rho v$$

- Assume $\sum_{j \in E} u_j = 1, \sum_{j \in E} u_j v_j = 1$, then u, v are unique.
- The other eigenvalue ρ' of A satisfies $|\rho'| < \rho$.
- Let $A^n = (a_{ij}^{(n)})$. Then

$$a_{ij}^{(n)} \sim \rho^n v_i u_j \ (n \to \infty).$$

Markov chain

- Let

$$p_{ij} = \frac{a_{ij}v_j}{\rho(A)v_i}.$$

Then (p_{ij}) is an irreducible stochastic matrix. Because

$$\sum_i u_i v_i p_{ij} = \frac{1}{\rho(A)} \sum_i u_i a_{ij} v_j = u_j v_j$$

and $uv = 1$, applying PF theorem, we know this stochastic matrix has limit distribution $\pi_i = u_i v_i$. By induction, we have

$$p_{ij}^{(n)} = \frac{a_{ij}^{(n)} v_j}{\rho(A)^n v_i}.$$

- Let $\mu_0 = \mu_n P^n$ with $\mu_n^{(i)}/x_n^{(i)} = \rho(A)^n v_i$. Then the collapse times for A and P are the same!

Brief proof

- Assume $\sum_i \mu_0^{(i)} = 1$, then

$$1 = \sum_i \mu_0^{(i)} = \sum_{ij} \mu_n^{(i)} p_{ij} = \sum_i \mu_n^{(i)}.$$

- If $\mu_n > 0$ for all n, then from the previous equation we see there exits a subsequence n_k such that $\mu_{n_k} \to \mu \geq 0$, and $\sum_i \mu^{(i)} = 1$.
- Thus

$$\mu_0 = \mu_{n_k} P^{n_k} \to (\mu \mathbf{1})\pi.$$

That is $\mu_0 = \pi$.

Some history

- L. K. Hua. The mathematical theory of global optimization on planned economy (in Chinese). Kexue Tongbao, (I): 1984, No.12, 705-709. (II) and (III): 1984, No.13, 769-772. (IV), (V), and (VI): 1984, No.16, 961-965. (VII): 1984, No.18, 1089-1092. (VIII): 1984, No.21, 1281-1282. (IX): 1985, No.1, 1-2. (X): 1985, No.9, 641-645. (XI): 1985, No.24, 1841-1844, 1984b.
- L. K. Hua and S. Hua. The study of the real square matrix with both left positive eigenvector and right positive eigenvector (in Chinese). Shuxue Tongbao, 8:30-32, 1985.
- M. F. Chen. Stochastic model of economic optimization (in Chinese). Chin. J. Appl. Probab. and Statis., (I): 8(3), 289-294; (II): 8(4), 374-377, 1992b.
- M. F. Chen and Y. Li. Stochastic model of economic optimization. J. Beijing Normal Univ., 30(2):185-194, 1994.

Chen's theorem: Stochastic model

- Assume that the non-negative random matrices $\{A_n, n \geq 1\}$ are IID.
- Stochastic model

$$x_0 = x_1 A_1 = x_2 A_2 A_1 = \cdots = x_n A_n A_{n-1} \cdots A_1.$$

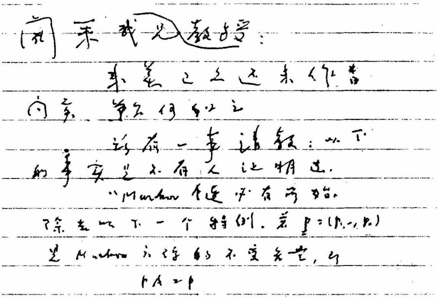
- Under some mild assumptions, the answer remains

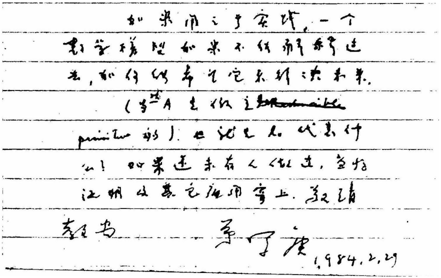
$$T = \inf\left\{n : \exists 1 \leq j \leq d, \quad x_n^{(j)} < 0\right\}$$

a.s. finite.

A Case

During his visit to CIT in 1984, Prof. Hua communicated to Prof K.L. Chung: "Markov chain must have a beginning."





Chung,Kai-Lai (1986) "Markov chain must have a beginning" In Memory of Prof. Loo-Keng Hua, J. Math. Res. Exposition (English Ed.), 6(1), 1-4.

Strong ergodicity

- Markov chain $P = (p_{ij})$ is called strongly ergodic if

$$||P - \pi||_{\text{Var}} := \sup_i \sum_j |p_{ij}^{(n)} - \pi_j| \to 0, \quad n \to \infty.$$

- Denote by $||x||_{\text{Var}}$ the total variance norm for a vector x:

$$||x||_{\text{Var}} = \sum_j |x_j|.$$

- Denote by $||M||_{\text{Var}}$ the total variance norm for a matrix M:

$$||M||_{\text{Var}} = \sup_i \sum_j |M_{ij}|.$$

- We have

$$||xM||_{\text{Var}} \le ||x||_{\text{Var}} ||M||_{\text{Var}}$$

An inequality

- Consider

$$\mu_0 = \mu_n P^n,$$

with $||\mu_0||_{\mathrm{Var}} = 1$.

- Assume P is strongly ergodic. Then

$$\mu_0 - \pi = \mu_n P^n - \pi = \mu_n (P^n - \pi),$$

so that

$$||\mu_0 - \pi||_{\mathrm{Var}} \leq ||\mu_n||_{\mathrm{Var}} ||P^n - \pi||_{\mathrm{Var}}.$$

- So

$$||\mu_n||_{\mathrm{Var}} \geq \frac{||\mu_0 - \pi||_{\mathrm{Var}}}{||P^n - \pi||_{\mathrm{Var}}} \to \infty$$

provided $x_0 \neq \pi$.

New equivalent collapse time

- Recall that

$$T := \inf\{n \geq 1 : \text{ there exists some } j \text{ such that } \mu_n^{(j)} < 0\}$$

- Assume $\sum_j x_0^{(j)} = 1 \Rightarrow \sum_j x_0^{(j)} = \sum_{ij} x_n^{(i)} p_{ij}^{(n)} = \sum_i x_n^{(i)} = 1.$
- Whenever $\mu_n^{(j)} < 0$ for some j, we have

$$||\mu_n||_{\mathrm{Var}} > 1.$$

- Because $|a| = a + 2a^-$, we have

$$\begin{aligned} ||\mu_n||_{\mathrm{Var}} &= \sum_j |\mu_n^{(j)}| = \sum_j \left(\mu_n^{(j)} + 2\left[\mu_n^{(j)}\right]^-\right) \\ &= 1 + 2\sum_j \left[\mu_n^{(j)}\right]^- \\ &> 1. \end{aligned}$$

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Collapse, Great recession(2008), or Great depression(1930's)

- Geometric convergence rate: $\exists C < \infty$ and $\lambda < 1$, such that

$$||P - \pi||_{\mathrm{Var}} \le C\lambda^n.$$

- Then

$$||\mu_n||_{\mathrm{Var}} \ge \lambda^{-n} \left[||\mu_0 - \pi||_{\mathrm{Var}}/C \right].$$

- So that the collapse time

$$T \le 1 + \log\left(||\mu_0 - \pi||_{\mathrm{Var}}/C \right) / \log \lambda.$$

- This argument can be applied to the stochastic model.

Value vs price

- v is value or the natural price. This was proved by Prof. Hua.
- Assume q_i is the price per i^{th} product. Then the cost per i^{th} product is
$$\sum_j a_{ij} q_j.$$
- Assume again the price of each product is in proportion, at the same ratio of λ:
$$\lambda q_i = \sum_j a_{ij} q_j, \quad \text{or} \quad \lambda q = Aq.$$
- By PF theorem, we must have
$$\lambda = \rho(A), \quad q = v.$$
- In this case, v_i is called the VALUE of i^{th} product, rather than the price.

Dimension

- Structure matrix a_{ij}:
$$\frac{j^{th}\text{dim}}{i^{th}\text{dim}};$$
- v is value:
$$v_i : \frac{\text{value}}{i^{th}\text{dim}};$$
-
$$a_{ij}v_j : \frac{j^{th}\text{dim}}{i^{th}\text{dim}} \times \frac{\text{value}}{j^{th}\text{dim}} = \frac{\text{value}}{i^{th}\text{dim}}, \quad \sum_j a_{ij}v_j : \frac{\text{(gross) value}}{i^{th}\text{dim}};$$
-
$$u_i a_{ij} v_j : i^{th}\text{dim} \times \frac{j^{th}\text{dim}}{i^{th}\text{dim}} \times \frac{\text{value}}{j^{th}\text{dim}} = \text{value}, \quad \sum_{ij} = \text{(gross) value};$$
- P is dimensionless:
$$p_{ij} = \frac{a_{ij}v_j}{\rho v_i} : \frac{\text{value}/i^{th}\text{dim}}{\text{value}/i^{th}\text{dim}}.$$

Variational formula

- Collatz(1942)-Wielandt(1950): for non-negative A,

$$\rho(A) = \min_{x>0} \max_i \frac{xA(i)}{x_i} = \max_{x>0} \min_i \frac{xA(i)}{x_i}.$$

- We see a dual of min-max and max-min variational formula for the PF eigenvalue $\rho(A)$. Therefore, we would like to call $\rho(A)^{-1}$ the "equilibrium rate of development".
- Doyle (2009):

$$\rho(A) = \max_{\mu>0} \min_{\phi_i\xi_i=\mu_i} \frac{\phi A\xi}{\phi\xi}.$$

- According the dimension as above,

$$\phi\xi : \text{output gross value}, \quad \phi A\xi : \text{input gross value}.$$

Symmetrization

- No symmetric A, that is, $A \neq A^*$, because of the dimensions.
- But we do have symmetrizable A, as that is the case for Prof. Hua's basic model concerning industry and agriculture. The 2×2 matrix is always symmetric.
- A is symmetric w.r.t. μ, that is $\mu_i a_{ij} = \mu_j a_{ji}$:

$$[i^{th}\text{dim}]^2 \times \frac{j^{th}\text{dim}}{i^{th}\text{dim}} = [j^{th}\text{dim}]^2 \times \frac{i^{th}\text{dim}}{j^{th}\text{dim}} = [i^{th}\text{dim}] \times [j^{th}\text{dim}];$$

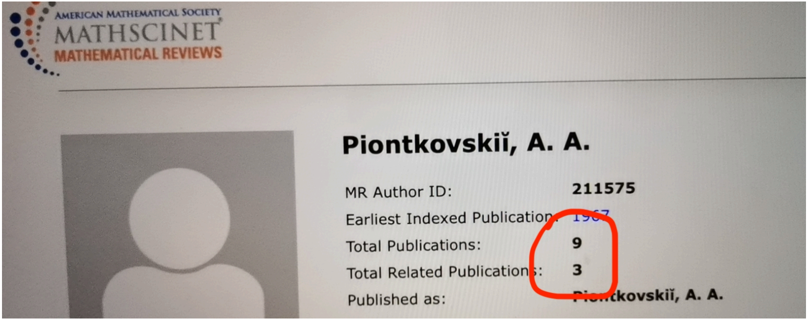
- or nondimensionaliz:

$$\sqrt{\frac{\mu_i}{\mu_j}} a_{ij} = \sqrt{\frac{\mu_j}{\mu_i}} a_{ji} : \sqrt{\frac{i^{th}\text{dim}^2}{j^{th}\text{dim}^2}} \times \frac{j^{th}\text{dim}}{i^{th}\text{dim}}.$$

Further reading

- Hua, L.-K.(1984). On the mathematical theory of globally optimal planned economic systems, Proc. Nati. Acad. Sci. USA 81, 6549-6553,
- Li, Bangxi (2017), Linear Theory of Fixed Capital and China's Economy – Marx, Sraffa and Okishio, Springer.
- Li, Bangxi, Zhao, Yihan and Fujimori, Yoriaki (2018), Marx-Okishio System and Perron-Frobenius Theorem, Post Keynesian Review, 6(1): 15-22.
- Morishima, M. (1973), Marx's Economics: A Dual Theory of Value and Growth, Cambridge University Press.
- Okishio, N. (1963), A Mathematical Note on Marxian Theorems, Weltwirtschaftliches Archiv, 1963, 91, 287-299

Some words

- Hua, Loo Keng (PRC-ASBJ) On the mathematical theory of globally optimal planned economic systems. Proc. Nat. Acad. Sci. U.S.A. 81 (1984), 20, Phys. Sci., 6549-6553.
- It seems, however, that any practically meaningful conclusions and recommendations either for planned or for market economies may be obtained only by using more sophisticated dynamic models.
- 

QED

THANKS !